MAXIMALLY SUPERSYMMETRIC SOLUTIONS OF TEN- AND ELEVEN-DIMENSIONAL SUPERGRAVITIES

JOSÉ FIGUEROA-O'FARRILL AND GEORGE PAPADOPOULOS

Abstract. We classify (up to local isometry) the maximally supersymmetric solutions of the eleven- and ten-dimensional supergravity theories. We find that the AdS solutions, the Hpp-waves and the flat space solutions exhaust them.

Contents

1. Introduction and main results

The investigation of string solitons, such as branes, has ushered in an era of rapid progress in our understanding of nonperturbative string theory. At low energies these solitons are described by (often supersymmetric) solutions of the corresponding supergravity theory. Conversely, it is believed that these supergravity solutions can be lifted

Date: November 13, 2002.
EMPG-02-16.

to solutions of the full string theory equations of motion, including all α' corrections. Equally relevant are the supersymmetric solutions of eleven-dimensional supergravity, as this is the low-energy limit of M-theory, itself the strong coupling limit of type IIA string theory.

Despite the huge catalogue of supersymmetric solutions of supergravity theories, we are still far from possessing an overall picture of the moduli space of such solutions and in fact until recently some basic questions remained unanswered.

The most important invariant of a supergravity background is the amount of supersymmetry that it preserves, usually labeled as a fraction, traditionally denoted ν, of the supersymmetry of the vacuum. In eleven-dimensional and type II supergravity theories, this is a fraction taking values in the set $\{0, \frac{1}{32}, \frac{1}{16}, \dots, \frac{31}{32}, 1\}$. At the time of writing it is now known whether all such fractions can in fact occur, although some progress has been made recently in the construction of backgrounds possessing hitherto unseen fractions $\frac{1}{2} < \nu < 1$.

This fraction is defined as follows. We are only concerned with solutions of the equations of motion where the fermionic fields are set to zero. Such a solution is supersymmetric if it is left invariant under some nontrivial supersymmetry transformations. These transformations are parametrised by spinors and, since the fermions have been put to zero, the only nontrivial transformations are those of the fermions themselves. The supersymmetric variation of the gravitino Ψ_M defines a covariant derivative $\mathcal{D}$ which is induced from a connection on the bundle of spinors:

$$\delta_\varepsilon \Psi_M = \mathcal{D}_M \varepsilon \ . \tag{1}$$

The tensor which measures the deviation of $\mathcal{D}$ from the spin connection ∇ depends algebraically on the bosonic fields of the theory. The other fermionic fields (if any) give rise to algebraic equations of the form, say,

$$\delta_\varepsilon \psi = \mathcal{A} \varepsilon \ , \tag{2}$$

where $\mathcal{A}$ is algebraic (i.e., zeroth order differential operator), itself depending on the bosonic fields of the supergravity theory. A (real) spinor ε is called a *Killing spinor* if it obeys the above equations, keeping in mind that the second equation may not arise if there are no other fermionic fields beside the gravitino—this happens for example, in eleven-dimensional supergravity.

Because the above equations are linear, the Killing spinors form a vector space whose dimension is at most the rank of the spinor bundle $\mathcal{S}$ of which the supersymmetry parameters ε are sections, which for the theories under consideration will be either 32 or 16. The reason is that equation (1) says that a Killing spinor is covariant constant with respect to the connection $\mathcal{D}$ and hence parallel transport uniquely defines a Killing spinor on all points of the spacetime from its value at any given point.

Finally, the fraction ν is defined as the following ratio

$$\nu = \frac{\dim\{\text{Killing spinors}\}}{\operatorname{rank} \mathcal{S}} \ .$$

Equation (1) allows us to define another invariant of the solution which refines the fraction ν, namely the holonomy representation of the connection $\mathcal{D}$. This is a refinement of the fraction ν because one can recover ν from the dimension of the subspace of invariants in the spinor representation (subject perhaps to the additional algebraic equations), and because different holonomy representations actually give rise to the same fraction. If we turn off all fields but the metric, so that we consider a purely gravitational solution, $\mathcal{D}$ coincides with the spin connection ∇ whose holonomy group is contained in (the spin cover of) the Lorentz group and, in the case of a supersymmetric background, must also be contained in the isotropy group of a nonzero spinor. As shown in [?, ?] for the case of eleven-dimensional supergravity, this can either be $\mathrm{SU}(5)$ or $\mathbb{R}^9 \rtimes \mathrm{Spin}(7)$. In the former case this leads to static spacetimes generalising the Kaluza–Klein monopole [?, ?, ?], whereas the latter case corresponds to generalisations of the purely gravitational pp-wave [?] which involve lorentzian holonomy groups acting reducibly yet indecomposably on Minkowski spacetime. In ten dimensions the situation is even simpler and the isotropy group of a chiral spinor must be contained in $\mathbb{R}^8 \rtimes \mathrm{Spin}(7)$ [?]. When we turn on the other fields in the background, the analysis of the connection $\mathcal{D}$ is complicated by the facts that $\mathcal{D}$ is not induced from a connection on the frame bundle and that its holonomy is generic. A holonomy analysis along the lines advocated in [?] has not yet been performed for supersymmetric solutions with flux. However much progress has been done in special cases [?, ?] leading to the no-go theorem of [?] for compactifications with torsion.

In this paper we take a first step in this direction by classifying, up to local isometry, all those solutions of eleven and ten-dimensional supergravity theories for which the (restricted) holonomy of $\mathcal{D}$ is trivial; in other words, we classify the maximally supersymmetric solutions or *vacua*.

With the exception of the massive IIA theory which, as we will see below, has no maximally supersymmetric background, every other supergravity theory in ten and eleven dimensions has a "trivial" vacuum in which the metric is flat and there are no fluxes. In addition, it has been known for some time that both eleven-dimensional and type IIB supergravities have vacua of the form $\mathrm{AdS}_{p+2} \times S^{D-p-2}$ [?, ?, ?] for $(D,p) \in \{(11,2),(11,5),(10,3)\}$.

It is now well known that given any solution of a supergravity theory, its plane wave limit [?, ?] yields another solution which, as proven in [?], preserves at least as much supersymmetry as the original solution. It follows that the plane wave limit of a maximally supersymmetric solution will also be maximally supersymmetric. Indeed it was shown in [?] that taking a plane wave limit of the $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ vacua of eleven-dimensional supergravity one recovers the maximally supersymmetric plane wave discovered by Kowalski-Glikman [?, ?] and re-discovered in [?] where it was identified as an *Hpp-wave*; that is, a plane wave whose geometry is that of a lorentzian symmetric space [?] and whose fluxes are homogeneous. Similarly, it was also shown in [?], that a plane wave limit of the $\mathrm{AdS}_5 \times S^5$ vacuum of type IIB supergravity yields the recently discovered maximally supersymmetric Hpp-wave solution [?]. Furthermore, it was also shown in [?] that other plane wave limits yield either the flat vacuum or again the known maximally supersymmetric Hpp-waves, so that no further vacua are obtained in this way. In this paper we will prove that there are no further maximally supersymmetric solutions in any of the eleven- and ten-dimensional supergravity theories.[1] Our proof consists of a systematic investigation of the conditions which follow from demanding that the curvature of the connection $\mathcal{D}$ vanishes. This analysis will in fact allow us to (re)derive the existence of all the known maximally supersymmetric solutions. In particular we will derive the Freund–Rubin ansatz from maximal supersymmetry. It is perhaps remarkable that it is actually possible to characterise these conditions geometrically and hence determine exactly all the solutions.

In addition to being a first step in the classification programme of supersymmetric backgrounds, the study of vacua is physically interesting since every vacuum defines a different stable sector of the theory in which to study excitations, both perturbative and solitonic. This has been extensively studied for the flat vacuum and to some extent for the AdS vacua, but in principle all vacua are to be treated on the same footing. Moreover the existence of the Hpp-wave vacua and in particular their interpretation as plane wave limits, has led to new progress on the AdS/CFT correspondence [?] and in particular has led to a gauge-theoretic derivation [?] of the spectrum of the IIB superstring on flat space and on the Hpp-wave, which can be quantized exactly [?, ?] in the light-cone gauge.

It is natural to wonder whether it is possible to extend this programme to solutions which preserve less than maximal supersymmetry, as there are several classes of such solutions that may have applications in string theory. For example, strings in Hpp-waves can be quantised exactly in the light-cone gauge and give rise to free massive theories in two dimensions[?]. More generally, strings on a larger class of pp-waves give rise to interacting massive theories in two dimensions [?]. The Hpp-wave ansatz has also proven useful in constructing solutions

[1] The AdS solution of type I supergravity found in [?] is erroneously claimed to be maximally supersymmetric when in fact it preserves only one-half of the supersymmetry.

with "exotic" fractions of supersymmetry: although the generic Hpp-wave solution preserves one half of the supersymmetry, there are points in the moduli space of Hpp-waves besides the one corresponding to the maximally supersymmetric solution, where $\frac{1}{2} < \nu < 1$ [?, ?, ?]. Another interesting class of solutions are those that preserve half of the supersymmetry and so include all elementary brane solutions as well as certain non-threshold bound states and other solutions in their U-duality orbit. The catalogue of such solutions is certainly large and a classification is not known, but we believe that the systematic approach advocated in this paper will prove useful in providing, if not a classification, at least a geometric characterisation of such solutions.

Our approach to the determination of the maximally supersymmetric solutions is based on the following strategy. Maximal supersymmetry implies the flatness of the connection $\mathcal{D}$, which becomes an algebraic equation

$$[\mathcal{D}_M, \mathcal{D}_N] = \mathcal{R}_{MN} = 0 \;, \tag{3}$$

which is supplemented (in some cases) by the algebraic equation $\mathcal{A} = 0$, derived from (2). The operators $\mathcal{R}_{MN}$ and $\mathcal{A}$ are endomorphisms of the spinor representation which can be expanded in terms of antisymmetric products of gamma matrices of the corresponding Clifford algebra. Due to linear independence, the coefficients of each antisymmetric product of gamma matrices must vanish separately. This gives a set of algebraic equations which we will analyse and solve yielding the results we summarise below.

Before we outline the results of the paper, we need to introduce some notation. Throughout this paper we will use the notation $\mathrm{CW}_D(A)$ to denote the D-dimensional lorentzian symmetric space with metric

$$g = 2dx^+dx^- + \left(\sum_{i,j=1}^{D-2} A_{ij}x^ix^j\right)(dx^-)^2 + \sum_{i=1}^{D-2}\left(dx^i\right)^2 \;, \tag{4}$$

where $A = [A_{ij}]$ is a constant symmetric matrix. More details on these spaces can be found in [?, ?] and a concrete isometric embedding in $\mathbb{E}^{2,D}$ can be found in [?]. Similarly we will use the notation $\mathrm{AdS}_D(R)$ and $S^D(R)$ to denote the D-dimensional anti de Sitter spacetime and the D-dimensional sphere, respectively, where R stands for the value of the scalar curvature.

Let us now describe the results of this paper. In Section 2 we will identify the vacua of eleven-dimensional supergravity up to local isometry and prove the following:

Theorem 1. *Let (M, g, F_4) be a maximally supersymmetric solution of eleven-dimensional supergravity. Then it is locally isometric to one of the following:*

- $\mathrm{AdS}_7(-7R) \times S^4(8R)$ *and* $F = \sqrt{6R}\,\mathrm{dvol}(S^4)$*, where* $R > 0$ *is the constant scalar curvature of* M*;*
- $\mathrm{AdS}_4(8R) \times S^7(-7R)$ *and* $F = \sqrt{-6R}\,\mathrm{dvol}(\mathrm{AdS}_4)$*, where* $R < 0$ *is again the constant scalar curvature of* M*; or*
- $\mathrm{CW}_{11}(A)$ *with* $A = -\frac{\mu^2}{36}\,\mathrm{diag}(4,4,4,1,1,1,1,1,1)$ *and* $F = \mu\, dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$*. One must distinguish between two cases:*
  - $\mu = 0$*: which recovers the flat space solution* $\mathbb{E}^{1,10}$ *with* $F = 0$*; and*
  - $\mu \neq 0$*: all these are isometric and describe an Hpp-wave.*

As a corollary we will determine the vacua of type IIA supergravity and prove the following:

Theorem 2. *Any maximally supersymmetric solution of type IIA supergravity is locally isometric to* $\mathbb{E}^{1,9}$ *with zero fluxes and constant dilaton.*

Theorems 1 and 2 were announced in [**?**].

In Section 3 we will determine the maximally supersymmetric solutions of ten-dimensional type IIB supergravity and prove the following result:

Theorem 3. *Let* $(M, g, F_5^+, \ldots)$ *be a maximally supersymmetric solution of ten-dimensional type IIB supergravity. Then it has constant axion and dilaton (normalised to 0 in the formulas below), all fluxes vanish except for the one corresponding to the self-dual five-form, and is locally isometric to one of the following:*

- $\mathrm{AdS}_5(-R) \times S^5(R)$ *and* $F = 2\sqrt{\frac{R}{5}}\,(\mathrm{dvol}(\mathrm{AdS}_5) + \mathrm{dvol}(S^5))$*, where* $\pm R$ *are the scalar curvatures of* AdS_5 *and* S^5*, respectively; or*
- $\mathrm{CW}_{10}(A)$ *with* $A = -\mu^2 \mathbb{1}$ *and* $F = \frac{1}{2}\mu\, dx^- \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8)$*. Again one must distinguish between two cases:*
  - $\mu = 0$*: which yields the flat space solution* $\mathbb{E}^{1,9}$ *with zero fluxes; and*
  - $\mu \neq 0$*: all these are isometric and describe an Hpp-wave.*

The proof of this theorem rests on a conjectured Plücker-style formula for orthogonal planes which we have verified for the case at hand, and which we believe to hold in more generality. The more general conjecture as well as its verification in some cases is presented in a separate article [**?**].

Finally in Section 4 we will prove the following two results on the remaining ten-dimensional supergravities:

Theorem 4. *Any maximally supersymmetric solution of type I or heterotic ten-dimensional supergravity is locally isometric to flat space with zero fluxes and constant dilaton.*

Theorem 5. *Massive IIA supergravity has no maximally supersymmetric solutions.*

As a corollary of the determination of the maximally supersymmetric solutions of ten- and eleven-dimensional supergravity theories, we can determine all the maximally supersymmetric solutions of lower dimensional supergravity theories which are obtained as toroidal reductions from them, or more generally as quotients by a group action. In fact, it is not hard to show, using the methods in [**?**], that the only vacua for these theories are flat with vanishing fluxes and are obtained by quotienting the flat vacua in ten and eleven dimensions by a translation subgroup of the isometries.

2. Maximal supersymmetry in eleven-dimensional supergravity

2.1. **Eleven-dimensional supergravity.** The bosonic part of the action of eleven-dimensional supergravity [**?**, **?**] is

$$\int_M \left(\tfrac{1}{2}R\,\mathrm{dvol} - \tfrac{1}{4}F\wedge\star F + \tfrac{1}{12}F\wedge F\wedge A\right)\ , \tag{5}$$

where $F = dA$ is the four-form field strength, R is the scalar curvature of the metric g and dvol is the (signed) volume element

$$\mathrm{dvol} := \sqrt{|g|}\,dx^0\wedge dx^1\wedge\cdots\wedge dx^{10}\ .$$

The associated field equations of this action are as follows:

$$\begin{aligned} d\star F &= -\tfrac{1}{2}F\wedge F\\ R_{MN} - \tfrac{1}{2}g_{MN}R &= \tfrac{1}{2}F^2_{MN} - \tfrac{1}{4}g_{MN}F^2\ , \end{aligned} \tag{6}$$

where we have defined the partial contractions

$$F^2_{MN} = \tfrac{1}{6}F_{MPQR}F_N{}^{PQR} \qquad \text{and} \qquad F^2 = \tfrac{1}{24}F_{MNPQ}F^{MNPQ}\ ,$$

and where

$$R_{MN} = R_{MPN}{}^P \qquad \text{and} \qquad [\nabla_M,\nabla_P]X^N = R_{MP}{}^N{}_Q X^Q\ ,$$

with ∇ the Levi-Civita connection of g, R the curvature of ∇ and $M, N, \dots = 0,\dots,10$. Notice that taking the trace of the Einstein-type equation we obtain that

$$R = \tfrac{1}{6}F^2\ .$$

In order to discuss supersymmetric solutions we have to say a word about spinors. We will be working with the Clifford algebra

$$\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = +2\eta_{AB}\mathbb{1}\ , \tag{7}$$

with η the *mostly plus* metric and the indices $A, B = 0,\dots,10$ being frame indices. In the standard notation (see, e.g., [**?**]) this defines the Clifford algebra $\mathrm{C}\ell(1,10)$ which is isomorphic to $\mathrm{Mat}(32,\mathbb{R})\oplus\mathrm{Mat}(32,\mathbb{R})$ and so it has two irreducible representations $S^\pm$ isomorphic to $\mathbb{R}^{32}$. These are distinguished by the action of the volume element $\Gamma^{12} = \pm\mathbb{1}$, respectively. Of course, both $S^{\pm}$ are isomorphic, as representations of $\mathrm{Spin}(1,10) \subset \mathrm{C}\ell(1,10)$, to the unique spinor representation S of $\mathrm{Spin}(1,10)$. In our conventions, the gravitino belongs to S^{-}.

The supersymmetric variation of the gravitino defines the so-called supercovariant connection

$$\mathcal{D}_M = \nabla_M - \tfrac{1}{288}\left(\Gamma^{PQRS}{}_M + 8\Gamma^{PQR}\delta^S_M\right) F_{PQRS} , \tag{8}$$

where the spin connection is given by

$$\nabla_M = \partial_M + \tfrac{1}{4}\omega_M{}^{AB}\Gamma_{AB} , \tag{9}$$

where we have used the same symbol ∇ to denote the Levi-Civita connection of g and the associated spin connection.

The supercovariant connection $\mathcal{D}$ is a connection of the bundle of spinors associated to the Clifford representation S^{-}, which is *not* induced from the frame bundle because of the term proportional to the four-form F.

A solution of eleven-dimensional supergravity is supersymmetric if it admits nonzero *Killing spinors* ϵ defined by

$$\mathcal{D}_M \epsilon = 0 . \tag{10}$$

The number of supersymmetries preserved by a solution is the maximum number of linearly independent Killing spinors. A solution is maximally supersymmetric if it has thirty-two linearly independent Killing spinors. This is equivalent to the connection $\mathcal{D}$ having trivial holonomy:

$$\mathrm{Hol}(\mathcal{D}) = 1 . \tag{11}$$

Iterating equation (10) leads to the integrability condition

$$[\mathcal{D}_M, \mathcal{D}_N]\epsilon = \mathcal{R}_{MN}\epsilon = 0 , \tag{12}$$

where $\mathcal{R}_{MN} = [\mathcal{D}_M, \mathcal{D}_N]$ is the curvature of the supercovariant connection. A necessary condition for maximal supersymmetry is the zero curvature condition:

$$\mathcal{R}_{MN} = 0 . \tag{13}$$

This equation can be expanded in a basis of the Clifford algebra (modulo the centre), which is given by the skew-symmetric products of gamma-matrices. The requirement that $\mathcal{R} = 0$ implies that every component of $\mathcal{R}$ in this basis should vanish.

The zero curvature condition is equivalent to the triviality of the *restricted* holonomy of the supercovariant connection: that is, the holonomy around contractible loops. Hence on a simply-connected spacetime, the zero curvature condition implies the holonomy condition (11). For a non-simply connected spacetime, the integrability condition (13) is necessary but not sufficient. In addition one must ensure that the Killing spinors are either periodic or antiperiodic along noncontractible loops. In this paper we will focus on local solutions, and this distinction will not arise.

These necessary and sufficient conditions for the existence of a maximally supersymmetric spacetime can be easily extended to other supergravities. In the ten-dimensional supergravities treated in this paper, the Killing spinor equation (10) is supplemented by algebraic conditions coming from the supersymmetry variation of fermionic fields other than the gravitino which are present in the theory. Therefore to find the solutions with maximal supersymmetry, these algebraic Killing spinor equations have to be imposed in addition to the vanishing of the curvature of the supercovariant connection. The additional algebraic Killing spinor equations do not alter the above discussion of the existence of Killing spinors in non-simply connected spacetimes.

2.2. **The zero curvature equations.** In this section we will study the equations which arise from demanding the vanishing of the curvature $\mathcal{R}$ of the supercovariant connection $\mathcal{D}$. After some computation we find that (cf. [?])

$$\begin{aligned}
\mathcal{R}_{MN} = \tfrac{1}{4}R_{MNAB}\Gamma^{AB} &- \frac{2}{(288)^2}F_{M_1\dots M_4}F_{N_1\dots N_4}\epsilon_{MN}{}^{M_1\dots M_4N_1\dots N_4}{}_L\Gamma^L \\
&+ \tfrac{48}{(288)^2}\big[4F_{MLPQ}F^{LPQ}{}_{M_1}\Gamma^{M_1}{}_N - 4F_{NLPQ}F^{LPQ}{}_{M_1}\Gamma^{M_1}{}_M \\
&- 36F_{LPMM_1}F^{LP}{}_{NN_1}\Gamma^{M_1N_1} + F_{L_1\dots L_4}F^{L_1\dots L_4}\Gamma_{MN}\big] \\
&+ \tfrac{1}{36}\big[\nabla_M F_{NM_1M_2M_3} - \nabla_N F_{MM_1M_2M_3}\big]\Gamma^{M_1M_2M_3} \\
+ \tfrac{8}{(288)^2 3}\big[F_{M_1\dots M_4}&F_{N_1\dots N_3N}\epsilon_M{}^{M_1\dots M_4N_1\dots N_3}{}_{L_1\dots L_3} - (N\leftrightarrow M)\big]\Gamma^{L_1L_2L_3} \\
&- \tfrac{1}{432}\big[4F_{LM_1\dots M_3}F^L_{MNN_1}\Gamma^{M_1\dots M_3N_1} \\
+ 3F_{LPM_1M_2}F^{LPN_1}{}_N&\Gamma^{M_1M_2}{}_{MN_1} - 3F_{LPM_1M_2}F^{PLN_1}{}_M\Gamma^{M_1M_2}{}_{NN_1}\big] \\
&- \tfrac{1}{288}\big[\nabla_M F_{N_1\dots N_4}\Gamma^{N_1\dots N_4}{}_N - (N\leftrightarrow M)\big] \\
&+ \tfrac{1}{(72)^2 5!}\big[-6F_{MM_1\dots M_3}F_{NN_1\dots N_3}\epsilon^{M_1\dots M_3N_1\dots N_3}{}_{L_1\dots L_5} \\
&- 6F_{MPM_1M_2}F^P_{N_1\dots N_3}\epsilon_N{}^{M_1M_2N_1\dots N_3}{}_{L_1\dots L_5} \\
&+ 6F_{NPM_1M_2}F^P_{N_1\dots N_3}\epsilon_M{}^{M_1M_2N_1\dots N_3}{}_{L_1\dots L_5} \\
&+ 9F_{PQM_1M_2}F^{PQ}{}_{N_1N_2}\epsilon_{MN}{}^{M_1M_2N_1N_2}{}_{L_1\dots L_5}\big]\Gamma^{L_1\dots L_5}\ , \qquad (14)
\end{aligned}$$

where we have used that

$$\Gamma^{A_1\dots A_{2k}} = \tfrac{(-1)^k}{(11-2k)!}\epsilon^{A_1\dots A_{2k}}{}_{B_1\dots B_{11-2k}}\Gamma^{B_1\dots B_{11-2k}}\ . \qquad (15)$$

Maximal supersymmetry demands that the coefficient of every term in the above expansion of $\mathcal{R}$ in skew-symmetric products of the gamma-matrices should vanish. This gives an over-determined system of equations which we now analyse in turn.

2.2.1. *Terms linear in* Γ. The vanishing of the term in the curvature of the supercovariant connection linear in the Γ-matrices implies that

$$F \wedge F = 0 \ . \tag{16}$$

Taking the inner product of this condition with respect to a vector field X, we find that

$$\iota_X F \wedge F = 0 \ . \tag{17}$$

2.2.2. *Terms quadratic in* Γ. The vanishing of the term in the curvature of the supercovariant connection quadratic in the Γ-matrices implies the following equation:

$$\begin{aligned} R_{MNPQ} &+ \tfrac{1}{36}\left(F^2_{NP}g_{MQ} - F^2_{MP}g_{NQ} - F^2_{NQ}g_{MP} + F^2_{MQ}g_{NP}\right) \\ &- \tfrac{1}{12}\left(F^2_{MPNQ} - F^2_{MQNP}\right) + \tfrac{1}{36}F^2\left(g_{MP}g_{NQ} - g_{MQ}g_{NP}\right) = 0 \ , \end{aligned} \tag{18}$$

where we have introduced the partial contraction

$$F^2_{MNPQ} = \tfrac{1}{2}F_{MNRS}F_{PQ}{}^{RS} \ .$$

Tracing the above equation in N, Q we recover the Einstein field equations

$$R_{MN} = \tfrac{1}{2}F^2_{MN} - \tfrac{1}{6}g_{MN}F^2 = 0 \ , \tag{19}$$

and in particular a relation between the Ricci scalar and the norm of the four-form:

$$R = \tfrac{1}{6}F^2 \ . \tag{20}$$

It is clear from (18) that if $F = 0$ the curvature of the spacetime vanishes. Thus the only such solution is locally isometric to Minkowski spacetime.

2.2.3. *Terms cubic in* Γ. The vanishing of the component of $\mathcal{R}$ cubic in Γ is

$$\begin{aligned} &\left(\nabla_M F_{NL_1L_2L_3} - \nabla_N F_{ML_1L_2L_3}\right) \\ &- \tfrac{1}{816}\left(F_{M_1\dots M_4}F_{N_1\dots N_3 N}\epsilon_M{}^{M_1\dots M_4 N_1\dots N_3}{}_{L_1\dots L_3} - (N \leftrightarrow M)\right) = 0 \ . \end{aligned} \tag{21}$$

This equation can be simplified using equation (17). Indeed, in components (17) can be written as

$$F_{M[L_1L_2L_3}F_{L_4\dots L_7]} = 0 \ . \tag{22}$$

Substituting this equation back into (21), we find that

$$\nabla_M F_{NL_1L_2L_3} - \nabla_N F_{ML_1L_2L_3} = 0 \ . \tag{23}$$

This together with the fact that F is a closed four-form imply it is parallel:

$$\nabla F = 0 \ . \tag{24}$$

Observe that the above equation and (16) imply the field equations (6) of the four-form field strength.

Equation (18) expresses the Riemann curvature tensor algebraically in terms of F and g, both of which are parallel with respect to the Levi-Civita connection. This means that the Riemann curvature tensor is also parallel, and we conclude that a maximally supersymmetric solution of eleven-dimensional supergravity is locally symmetric. Moreover equation (24) says that the four-form F is invariant.

2.2.4. *Terms quartic in* Γ. The vanishing of the component of $\mathcal{R}$ fourth order Γ is

$$2F_{L[M_1M_2M_3}F^{MNL}{}_{M_4]} - 3F_{LP[M_1M_2}F^{LP}{}_{M_3}{}^{[N}\delta^{M]}{}_{M_4]} = 0\ . \tag{25}$$

Antisymmetrising in all free indices, we find

$$F_{L[M_1M_2M_3}F^{L}{}_{M_4M_5M_6]} = 0\ . \tag{26}$$

Antisymmetrising in five of the six free indices in (25), we find

$$2F_{L[M_1M_2M_3}F^{LM}{}_{M_4M_5]} - \tfrac{3}{2}F_{LP[M_1M_2}F^{LP}{}_{M_3M_4}\delta^{M}{}_{M_5]} = 0\ . \tag{27}$$

Next we contract the indices M and M_4 in (25) to find

$$F_{LP[M_1M_2}F^{LP}{}_{M_3]}{}^{N} = 0\ . \tag{28}$$

This in turn implies that

$$F_{LP[M_1M_2}F^{LP}{}_{M_3M_4]} = 0\ . \tag{29}$$

2.2.5. *Terms quintic in* Γ. Let us next investigate the conditions that arise from the vanishing of the fifth order terms in Γ. These are

$$\begin{aligned} &-2F^{M}{}_{[P_1P_2P_3}F^{N}{}_{Q_1Q_2Q_3]} - 2F^{M}{}_{L[P_2P_3}\delta^{N}{}_{P_1}F^{L}{}_{Q_1Q_2Q_3]} \\ &+2F^{N}{}_{L[P_2P_3}\delta^{M}{}_{P_1}F^{L}{}_{Q_1Q_2Q_3]} + 3\delta^{MN}{}_{[P_1P_2}F_{|LP|P_3Q_1}F^{LP}{}_{Q_2Q_3]} = 0\ , \end{aligned} \tag{30}$$

where $\delta^{MN}_{PQ} = \delta^{[M}_{P}\delta^{N]}_{Q}$. Combining (25), (29) with (30), we find that

$$F_{M[P_1P_2P_3}F_{Q_1Q_2Q_3]N} = 0 \tag{31}$$

or equivalently

$$\iota_X F \wedge \iota_Y F = 0\ . \tag{32}$$

This concludes the investigation of the various conditions that arise from the vanishing of the curvature of the supercovariant connection.

2.3. F **is decomposable.** To analyse further the conditions that we have derived in the previous section, we shall use the *Plücker relations*. A p-form is said to be *decomposable* if it can be written as the wedge product of p one-forms. It is a classical result in algebraic geometry (see, for example, [**?**, Chapter 1]) that a p-form F is decomposable if and only if

$$\iota_\Xi F \wedge F = 0 \tag{33}$$

for every $(p-1)$-multivector Ξ. In this section we shall show that the conditions (17) and (32) derived in the previous section actually imply

(33) and hence that F is decomposable. Observe that the converse is trivially true: if F is decomposable then (17) and (32) are satisfied.

Our starting point is equation (17). Contracting this equation with another vector field Y, we obtain

$$\iota_Y\iota_X F \wedge F - \iota_X F \wedge \iota_Y F = \iota_Y\iota_X F \wedge F = 0 \ ,$$

where to establish the first equality we have used (32). Contracting the above equation with another vector field Z, we find

$$\iota_Z\iota_Y\iota_X F \wedge F = -\iota_Y\iota_X F \wedge \iota_Z F \ . \tag{34}$$

Next contracting equation (32) with a third vector field, we get

$$\iota_Y\iota_X F \wedge \iota_Z F = \iota_X F \wedge \iota_X F \wedge \iota_Y\iota_Z F = \iota_Y\iota_Z F \wedge \iota_X F \ . \tag{35}$$

Therefore the expression in the right-hand-side of (34) is symmetric in X and Z, whereas the left-hand-side of (34) is skew-symmetric. This means that both terms in (34) must vanish separately. In particular we find that

$$\iota_Z\iota_Y\iota_X F \wedge F = 0 \ , \tag{36}$$

which is precisely equation (33).

With all but equation (18) fully analysed we can already conclude that a solution (M, g, F) of eleven-dimensional supergravity is maximally supersymmetric if and only if (M, g) is a locally symmetric space and F is parallel and decomposable.

2.4. **The local geometries.** In this section we narrow down the possible choices of symmetric spaces that are maximally supersymmetric solutions of M-theory by exploiting the information that F is a parallel decomposable form in a (lorentzian) symmetric space. We will achieve a characterization of the geometry up to local isometry. First recall that if the four-form F vanishes, the only solution is Minkowski spacetime up to discrete identifications which preserve supersymmetry. In what follows we assume that $F \neq 0$.

We will be making use of the classification of lorentzian symmetric spaces by Cahen and Wallach [**?**]. They stated the following theorem:

Theorem 6. *Let (M, g) be a simply-connected lorentzian symmetric space. Then M is isometric to the product of a simply-connected riemannian symmetric space and one of the following:*

- $\mathbb{R}$ *with metric* $-dt^2$*;*
- *the simply-connected covering space of D-dimensional (anti) de Sitter space, where $D \geq 2$; or*
- *an Hpp-wave* $\mathrm{CW}_D(A)$ *with $D \geq 3$ and metric given by* (4)*.*

If we drop the hypothesis of simply-connectedness then this theorem holds up to local isometry. We will make use of this result repeatedly.

Let (M, g, F) be a maximally supersymmetric solution. As we have seen (M, g) is a locally symmetric space, whence locally isometric to one of the spaces in the list in the above theorem. Every such space is acted on transitively by a Lie group G (the group of *transvections*), whence if we fix a point in M (the *origin*) with isotropy H, M is isomorphic to the space of cosets G/H. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{h}$ the Lie subalgebra corresponding to H. Then $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is isomorphic to the tangent space of M at the origin. The Lie brackets are such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \ .$$

The metric g on M is determined by an $\mathfrak{h}$-invariant inner product B on $\mathfrak{m}$. Since the four-form F is parallel, it is G-invariant. This means that it is uniquely defined by its value at the origin, which defines an $\mathfrak{h}$-invariant four-form on $\mathfrak{m}$. Since it does not vanish (by hypothesis) and is decomposable, it determines a four-dimensional vector subspace $\mathfrak{n} \subset \mathfrak{m}$ as follows: if at the origin $F = \theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4$, then $\mathfrak{m}$ is the span of (the dual vectors to) the θ_i. Furthermore, because F is invariant, we have that H leaves the space $\mathfrak{n}$ invariant, whence $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$, which means that the holonomy group of M (which is isomorphic to H) acts reducibly. In lorentzian signature this does not imply that the space is locally isometric to a product, since the metric may be degenerate when restricted to $\mathfrak{n}$. Therefore we must distinguish between two cases, depending on whether or not the restriction $B|_{\mathfrak{n}}$ of B to $\mathfrak{n}$ is or is not degenerate.

If $B|_{\mathfrak{n}}$ is non-degenerate, then it follows from the de Rham–Wu decomposition theorem [?] that the space is locally isometric to a product $N \times P$, with N and P locally symmetric spaces of dimensions four and seven, respectively. Explicitly, we can see this as follows: there exists a B-orthogonal decomposition $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{p}$, with $\mathfrak{p} := \mathfrak{m}^{\perp}$, where $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ because of the invariance of the inner product. Let $\mathfrak{g}_N = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{g}_P = \mathfrak{h} \oplus \mathfrak{p}$. They are clearly both Lie subalgebras of $\mathfrak{g}$. Let G_N and G_P denote the respective (connected, simply-connected) Lie groups. Then N will be locally isometric to G_N/H and P will be locally isometric to G_P/H, and M will be locally isometric to the product. The metrics on N and P are induced by the restrictions of $\mathfrak{n}$ and $\mathfrak{p}$ respectively of the inner product B on $\mathfrak{n} \oplus \mathfrak{p}$, denoted

$$\begin{aligned} B_{\mathfrak{n}} &= B|_{\mathfrak{n}} \\ B_{\mathfrak{p}} &= B|_{\mathfrak{p}} \ . \end{aligned} \tag{37}$$

We shall denote the metrics on N and P induced from the above inner products by h and m, respectively.

On the other hand if the restriction $B|_{\mathfrak{n}}$ is degenerate, so that $\mathfrak{n}$ is a null four-dimensional subspace of $\mathfrak{m}$, the four-form F is also null. From Theorem 6 one sees (see, e.g., [?]) that the only lorentzian symmetric spaces admitting parallel null forms are those which are locally isometric to a product $M = \mathrm{CW}_d(A) \times Q_{11-d}$, where $\mathrm{CW}_d(A)$ is a d-dimensional Cahen-Wallach space and Q_{11-d} is an $(11-d)$-dimensional riemannian symmetric space.

In summary, there are two separate cases to consider:

1. $(M, g) = (N_4 \times P_7, h \oplus m)$ (locally), where (N, h) and (P, m) are symmetric spaces and where F is proportional to (the pull-back of) the volume form on (N, h); or
2. $M = \mathrm{CW}_d(A) \times Q_{11-d}$ (locally) and $d \geq 3$, where Q_{11-d} is a riemannian symmetric space.

The first case corresponds to the well-known Freund–Rubin Ansatz, which we have *derived* here from the requirement of maximal supersymmetry. The second case has also been considered before [**?**, **?**]. The Cahen–Wallach metrics are special cases of metrics admitting parallel null spinors [**?**, **?**]. This larger class of metrics have appeared in the supergravity literature, see for example [**?**].

We now investigate each of these two cases above separately.

2.4.1. *The Freund–Rubin Ansatz revisited.* We start by reconsidering the Freund–Rubin Ansatz: $(M, g) = (N_4 \times P_7, h \oplus m)$, locally. Since M is locally symmetric, we can analyze the equations implied by supersymmetry at the origin of the symmetric space. It is straightforward to see that the only non-trivial equation that remains to be solved is (18). Since the spacetime is isomorphic to a product, the curvature of spacetime decomposes into the curvatures of N and P.

Since F vanishes along P, the curvature of P is

$$R_{abcd} = -\tfrac{1}{3} R \left(m_{ac} m_{bd} - m_{ad} m_{bc} \right) \ , \tag{38}$$

where $a, b, c, d = 1, 2, \ldots, 7$ label the coordinates of P and R is the Ricci scalar of M. This shows that P is a space form. If $R < 0$ then P is locally isometric to S^7 with Ricci scalar $R_P = -7R$, whereas if $R > 0$, P is isometric to AdS_7 with Ricci scalar $-7R$.

Similarly the curvature of N is obtained by evaluating equation (18) along the directions of $\mathfrak{n}$. After some computation, we find that

$$R_{ijkl} = -\tfrac{2}{3} R \left(h_{ik} h_{jl} - h_{il} h_{jk} \right) \ . \tag{39}$$

Thus if $R < 0$, then N is locally isometric to AdS_4 with Ricci scalar $R_N = 8R$, whereas if $R > 0$, then N is locally isometric to S^4 with Ricci scalar $R_N = 8R$. Notice that $R_P + R_N = R$, as it should since R is the scalar curvature of $M = P \times Q$.

2.4.2. *The case of a null four-form.* If $F^2 = 0$, we have shown that $M = \mathrm{CW}_d(A) \times Q_{11-d}$, for $3 \leq d \leq 11$, where Q is a riemannian symmetric space. Since F is decomposable and null, it must be of the form

$$F = dx^- \wedge \varphi \ , \tag{40}$$

where dx^- is (up to scale) a parallel null 1-form, which exists in every $\mathrm{CW}_d(A)$, and φ is a parallel three-form on M with positive norm: $\varphi^2 > 0$.

Substituting (40) into the expression of the curvature tensor (18), we find that the curvature of Q vanishes. Therefore, M is locally isometric to $\mathrm{CW}_d(A) \times \mathbb{R}^{11-d}$.

The metric on $M = \mathrm{CW}_d(A) \times \mathbb{R}^{11-d}$ can be written in local coordinates as follows

$$ds^2 = 2dx^+dx^- + \sum_{i,j=1}^{9} A_{ij}x^i x^j (dx^-)^2 + \sum_{i=1}^{9} (dx^i)^2 \ .$$

where A is a symmetric 9×9 matrix which is degenerate along the $\mathbb{R}^{11-d}$ directions. In addition, we can always choose coordinates in $\mathbb{R}^9$ in such a way that the parallel, decomposable 4-form F is given by

$$F = \mu\, dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3 \ ,$$

where μ is some constant. As shown in [**?**] this is the ansatz for an Hpp-wave and as shown in that paper, the only maximally supersymmetric solution is the one in [**?**], for which $A = -\frac{\mu^2}{36}\mathrm{diag}(4,4,4,1,\ldots,1)$.

In summary, we have proven Theorem 1, stated in the introduction.

2.5. **Maximal supersymmetry in IIA supergravity.** Type IIA supergravity [**?**, **?**, **?**] is obtained by dimensional reduction from eleven-dimensional supergravity. This means that any solution of IIA supergravity can be uplifted (or oxidised) to a solution of eleven-dimensional supergravity possessing a one-parameter subgroup of the symmetry group such that reducing along its orbits yields the IIA solution we started out with. If the IIA supergravity solution preserves some supersymmetry, its lift to eleven dimensions will preserve at least the same amount of supersymmetry. This means that a maximally supersymmetric solution of IIA supergravity will uplift to one of the maximally supersymmetric solutions of eleven-dimensional supergravity determined in the previous section. Therefore the determination of the IIA vacua reduces to classifying those dimensional reductions of the eleven-dimensional vacua which preserve all supersymmetry.

As explained already in [**?**], the only such reductions are the reductions of the flat eleven-dimensional vacuum by a translation subgroup of the Poincaré group. This can also be verified by an explicit analysis of the Killing spinor equations of IIA supergravity.

In summary, this proves Theorem 2, stated in the introduction.

3. Maximal supersymmetry in IIB supergravity

The bosonic field content of IIB supergravity [**?**, **?**, **?**] is the metric g, two scalars, two complex three-form gauge potentials $\{A^1, A^2 : A^1 = (A^2)^*\}$ and a real four-form gauge potential A. The two scalars parametrise the upper half-plane $SU(1,1)/U(1)$, the two three-form gauge potentials $\{A^i; i=1,2\}$ transform as a $SU(1,1)$-doublet while A is a $SU(1,1)$-singlet. We follow mostly the notation of [**?**].

We shall not state the field equations of IIB supergravity here. This is because, as in the case of eleven-dimensional supergravity that we have already studied, the conditions for maximal supersymmetry derived from the Killing spinor equations imply all the field equations. To continue, we follow [**?**] and define the fields

$$\begin{aligned} P_M &= -\varepsilon_{ij} V_+^i \partial_M V_+^j \\ G_3 &= -\varepsilon_{ij} V_+^i F^j \\ F_5 &= dA + \tfrac{i}{16}\varepsilon_{ij} A^i \wedge F^j \end{aligned} \tag{41}$$

where $\{V_a^i : i=1,2, a=+,-\}$ are $SU(1,1)$ matrices parameterized by the two scalars which are $SU(1,1)$-doublets under rigid transformations. In addition V_+^i transforms with the standard one-dimensional complex representation on $U(1)$ under a local transformation and V_-^i transforms with its conjugate representation; $V_+^i = (V_-^i)*$. Moreover $F^i = dA^i$, F is a self-dual five-form and is a singlet under both rigid $SU(1,1)$ and local $U(1)$ transformations, and G is a singlet under rigid $SU(1,1)$ tranformations but transforms under the local $U(1)$ transformations.

Next introduce the canonical $U(1)$ connection

$$Q = -i\varepsilon_{ij} V_-^i dV_+^j \tag{42}$$

on the coset space $SU(1,1)/U(1)$. The Killing spinor equations of IIB supergravity are given by

$$\begin{aligned} \mathcal{D}_M\varepsilon = D_M\varepsilon &+ \tfrac{i}{192} F_{L_1\dots L_5}\Gamma^{L_1\dots L_5}\Gamma_M\varepsilon \\ &+ \tfrac{1}{96}\big(\Gamma_M{}^{L_1L_2L_3}\hat{G}_{L_1L_2L_3} - 9\Gamma^{L_1L_2}G_{ML_1L_2}\big)\varepsilon^* = 0 \\ \Gamma^M\varepsilon^* P_M - \tfrac{1}{24}&\Gamma^{MNR}G_{MNR}\varepsilon = 0\ , \end{aligned} \tag{43}$$

where

$$D_M = \nabla_M - \tfrac{i}{2}Q_M\ , \tag{44}$$

Q_M is the pull-back of the connection of $U(1)$ connection of the coset on to the spacetime, ε is a complex Weyl spinor, $\Gamma^{11}\varepsilon = \varepsilon$.

There are two types of Killing spinor equations for IIB supergravity. One is a parallel transport type of equation similar to that we have investigated in the context of eleven-dimensional supergravity. The other is an algebraic equation which does not involve derivatives on the spinor ε. Since we are seeking maximally supersymmetric solutions, the components of the algebraic Killing spinor equation as expanded in a basis of the Clifford algebra should vanish. This in particular implies that

$$P_M = 0 \qquad \text{and} \qquad G_{MNR} = 0\ . \tag{45}$$

Substituting the second equation above into the supercovariant derivative in (43), we find that it simplifies to

$$\mathcal{D}_M\varepsilon = D_M\varepsilon + \tfrac{i}{192}F_{L_1\ldots L_5}\Gamma^{L_1\ldots L_5}\Gamma_M\varepsilon\ . \tag{46}$$

The strategy that we shall adopt to find the maximally supersymmetric solutions is to compute the curvature $\mathcal{R}$ of the supercovariant derivative $\mathcal{D}$ above as we have done for eleven-dimensional supergravity. Indeed after some computation, we find that

$$\begin{aligned}\mathcal{R}_{MN} = \mathcal{F}_{MN} &+ \tfrac{1}{4}R_{MNPQ}\Gamma^{PQ} + \tfrac{i}{192}\nabla_M F_{NPQRS}\Gamma^{PQRS}\\ &- \tfrac{i}{192}\nabla_N F_{MPQRS}\Gamma^{PQRS} - \tfrac{1}{192}F_{ML_1L_2L_3P}F_N{}^{L_1L_2L_3}{}_Q\Gamma^{PQ}\\ &+ \tfrac{1}{12\cdot 64^2}F_{LMM_1M_2M_3}F^L{}_{NN_1N_2N_3}\varepsilon^{M_1M_2M_3N_1N_2N_3}{}_{PQRS}\Gamma^{PQRS}\ ,\end{aligned} \tag{47}$$

where $\mathcal{F}_{MN} = -\frac{i}{2}(\partial_M Q_N - \partial_N Q_M)$.

3.1. **The vanishing of the algebraic Killing spinor equation conditions.** Here we shall show that the conditions (45) of the algebraic Killing spinor equation required by maximal supersymmetry imply that the two scalars are constant and that $F^i = 0$. This is most easily seen by fixing the local U(1) symmetry of the coset space. This has been done in [**?**] and so we shall not repeat the computation here. After gauge fixing, the theory has two real scalars parameterized by the complex scalar B and a complex three three form field strength F. The final expressions for the relevant fields are the following:

$$\begin{aligned}G_{MNR} &= f(F_{MNR} - BF^*_{MNR})\\ P_M &= f^2\partial_M B\end{aligned} \tag{48}$$

where $f^{-2} = 1 - BB^*$, $BB^* < 1$. The condition $P_M = 0$ in (45) implies that the complex scalar field B is constant. The vanishing of $G = 0$ in (45) implies that $F_{MNR} = 0$. So the only fields that remain to be determined by maximal supersymmetry are the metric g and the self-dual five-form field strength F_5. In addition from (41) and (48), the five-form self-dual field strength is $F_5 = dA$. In particular, F_5 is closed and since it is self-dual, it is also co-closed.

Another consequence of the algebraic Killing spinor equation is that the pull-back of the U(1) curvature of the coset space $\mathrm{SU}(1,1)/\mathrm{U}(1)$ on the spacetime vanishes. This can be seen by the formula similar to those in (48) which expresses the pull-back U(1) connection in terms of B as

$$Q_M = f^2\mathrm{Im}\big(B\partial_M B^*\big) \tag{49}$$

Since B is constant, $Q_M = 0$.

3.2. **The vanishing of curvature conditions.**

3.2.1. *Terms zeroth order in* Γ. This term involves the pull-back of the curvature of the U(1) connection of coset space on the spacetime, ie

$$\mathcal{F}_{MN} = 0 \ . \tag{50}$$

This however vanishes as a consequence of the $Q_M = 0$ condition derived in the previous section from the algebraic Killing spinor equations. So there is no additional condition.

3.2.2. *Terms quadratic in* Γ. The condition on the quadratic terms in Γ is the following:

$$\tfrac{1}{4} R_{MNPQ} - \tfrac{1}{192} F_{ML_1L_2L_3[P} F_{|N|}{}^{L_1L_2L_3}{}_{Q]} = 0 \tag{51}$$

Contracting N and Q, we find the Einstein field equations

$$R_{MN} = \tfrac{1}{96} F_{ML_1L_2L_3L_4} F_N{}^{L_1L_2L_3L_4} \ . \tag{52}$$

In particular the Ricci scalar vanishes, $R = 0$, because F is self-dual.

3.2.3. *Terms fourth order in* Γ. The condition on the fourth order terms in Γ is the following:

$$\begin{aligned} &\tfrac{i}{192} \nabla_M F_{NPQRS} - \tfrac{i}{192} \nabla_N F_{MPQRS} \\ &\qquad + \tfrac{1}{12 \cdot 64^2} F_{LMM_1M_2M_3} F^L{}_{NN_1N_2N_3} \varepsilon^{M_1M_2M_3N_1N_2N_3}{}_{PQRS} = 0 \end{aligned} \tag{53}$$

In particular the imaginary and real parts of the above equation should vanish separately. Since F is real, this implies that

$$\nabla_M F_{NPQRS} - \nabla_N F_{MPQRS} = 0 \tag{54}$$

and

$$F_{LMM_1M_2M_3} F^L{}_{NN_1N_2N_3} \varepsilon^{M_1M_2M_3N_1N_2N_3}{}_{PQRS} = 0 \ . \tag{55}$$

Antisymmetrising (54) in the indices M, N, P, Q, R and using that $dF = 0$, we find that F is parallel with respect to the Levi-Civita connection. Combining this fact with equation (51) one concludes that the curvature tensor is parallel, whence (M, g) is locally symmetric.

The equation (55) can be rewritten in a more invariant form as

$$\iota_X F_L \wedge \iota_Y F^L = 0 \ . \tag{56}$$

Next contracting M with P in (55), the resulting equation can be written as

$$\iota_X F_L \wedge F^L = 0 \tag{57}$$

Taking the inner derivation of this equation with respect to the vector field Y and using the equation (56), we find that

$$\iota_Y \iota_X F_L \wedge F^L = 0 \ . \tag{58}$$

Now take the inner derivation of this equation with another vector field Z. This gives

$$\iota_Z \iota_Y \iota_X F_L \wedge F^L + \iota_Y \iota_X F_L \wedge \iota_Z F^L = 0 \ . \tag{59}$$

Taking the inner derivation of $\iota_X F_L \wedge \iota_Z F^L = 0$ with respect to Y, we find

$$\iota_Y \iota_X F_L \wedge \iota_Z F^L = \iota_Y \iota_Z F_L \wedge \iota_X F^L \tag{60}$$

which implies that the right hand side of (59) is symmetric in the interchange of X and Y while the left hand side is skew-symmetric. This implies that both terms in (59) should vanish separately. In particular, we have that

$$\iota_Z \iota_Y \iota_X F_L \wedge F^L = 0 \ . \tag{61}$$

This condition (61) is analogous to the Plücker relations which appear in eleven-dimensional supergravity. It is therefore conceivable that it should imply a decomposition of the self-dual five-form F. In fact, it can be shown that equation (61) implies that

$$F = G + \star G \ , \tag{62}$$

where G is a decomposable five-form. This is proven in [**?**], where we also state a conjectural generalisation involving p-forms in euclidean or lorentzian n-dimensional vector spaces, which we verify in low dimension. Notice that since F is parallel, then so is G.

3.3. **The local geometries.** We must distinguish between two cases, depending on whether or not the five-form G in equation (62) is null. First suppose that G (and hence F) is not null. An analysis similar to that presented for eleven-dimensional supergravity shows that the five-form G induces a local decomposition of (M, g) into a product $N_5 \times P_5$ of two five-dimensional symmetric spaces (N, h) and (P, m), where $G \propto \operatorname{dvol}(N)$ and hence $\star G \propto \operatorname{dvol}(P)$. Thus again we have derived the Freund–Rubin ansatz from maximal supersymmetry. Since (M, g) is lorentzian, one of the spaces (N, h) and (P, m) is lorentzian and the other riemannian. If the norm of G is positive, then N is riemannian and P is lorentzian, and vice-versa if G is negative. By interchanging G with $\star G$ if necessary, we can assume that G has positive norm and hence that N is riemannian.

We continue as in the eleven-dimensional case by analysing the remaining condition (51). Evaluating this along the directions of N_5, we find that

$$R_{ijkl} = \tfrac{1}{16} G^2 \left(h_{ik} h_{jl} - h_{jk} h_{il} \right) \ , \tag{63}$$

whence N is locally isometric to S^5. Similarly, evaluating equation (51) along P, we find that

$$R_{abcd} = -\tfrac{1}{16} G^2 \left(m_{ac} m_{bd} - m_{bc} m_{ad} \right) \ , \tag{64}$$

whence P is locally isometric to AdS^5. Both AdS_5 and S^5 have the same radii of curvature which are related to the norm of G in the following way: if S^5 has Ricci scalar R, then $G = 2\sqrt{\frac{R}{5}}\operatorname{dvol}(S^5)$.

Next suppose that G is null. As in the discussion of the eleven-dimensional case, and using the fact that the only symmetric spaces with null parallel forms are the CW Hpp-waves, the spacetime is locally isometric to $(M, g) = (\mathrm{CW}_d(A) \times Q_{10-d}, h \oplus m)$, where $d \geq 3$, and the five-form field strength is

$$F = dx^- \wedge \varphi \ , \tag{65}$$

where φ is a self-dual four-form and dx^- is the one-form dual to the parallel null vector in $\mathrm{CW}_d(A)$. Evaluating (51) along the directions of Q_{10-d}, we find that the curvature of Q_{10-d} vanishes because F is null. An analysis similar to that made for the eleven-dimensional case reveals that the metric and five-form of spacetime can be written as

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \sum_{i,j}^{8} A_{ij}x^ix^j(dx^-)^2 + \sum_{i=1}^{8}(dx^i)^2 \\ F_5 &= \mu dx^- \wedge (dx^{1234} + dx^{5678}) \ . \end{aligned} \tag{66}$$

This is precisely the ansatz used to find the maximally supersymmetric Hpp-wave solution of IIB supergravity [?]. Here we have derived it from the requirement of maximal supersymmetry. In particular, the maximally supersymmetric Hpp-wave solution occurs for $A = -\frac{\mu^2}{16}\mathrm{diag}(1, 1, \ldots, 1)$.

This concludes the proof of Theorem 3, stated in the introduction.

4. Maximal supersymmetry in other ten-dimensional supergravities

In this section we discuss the remaining ten-dimensional supergravity theories: the heterotic, type I and massive IIA supergravities.

4.1. **Heterotic supergravities.**

The only maximally supersymmetric solution of heterotic supergravities is the ten-dimensional Minkowski spacetime with constant dilaton and rest of form-field strengths to vanish. This can be easily seen by inspecting the Killing spinor equations

$$\begin{aligned} \nabla^+\varepsilon &= 0 \\ \Gamma^M\partial_M\phi\varepsilon - \tfrac{1}{12}H_{MNR}\Gamma^{MNR}\varepsilon &= 0 \\ F_{MN}\Gamma^{MN}\varepsilon &= 0 \ , \end{aligned} \tag{67}$$

where $\nabla^+ = \nabla + \frac{1}{2}H$, H is the NSNS three-form field strength, ϕ is the dilaton, ε is a Majorana-Weyl sixteen-component spinor and F is the curvature of the gauge connection with gauge group $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. (We have suppressed gauge indices). From the second Killing spinor equation one concludes that ϕ is constant and $H = 0$. The first equation implies that the curvature of the Levi-Civita connection vanishes and so the spacetime is locally isometric to Minkowski spacetime. The last Killing spinor equation implies that the curvature of the gauge fields vanishes and so for simply connected spacetimes the gauge connection vanishes as well. This proves the second part of Theorem 4 in the introduction.

4.2. **Type I supergravity.** Similarly, the only maximally supersymmetric solution of type I supergravity is Minkowski spacetime. This again can be easily seen by inspecting the Killing spinor equations

$$\begin{aligned} \nabla_M \eta + \tfrac{1}{8} H_{MNR} \Gamma^{NR} \eta &= 0 \\ \Gamma^M \partial_M \phi \eta + \tfrac{1}{12} H_{MNR} \Gamma^{MNR} \eta &= 0 \\ F_{MN} \Gamma^{MN} \eta &= 0 \ , \end{aligned} \tag{68}$$

where H is the RR three-form field strength, ϕ is the dilaton, ε is a Majorana-Weyl sixteen component spinor and F is the curvature of the gauge connection with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$. (We have suppressed gauge indices). Again the second Killing spinor equation implies that ϕ is constant and $H = 0$. Then the first implies that the curvature of the Levi-Civita connection vanishes and so the spacetime is locally isometric to Minkowski space. The last Killing spinor equation implies that the curvature of the gauge fields vanishes and so for simply connected spacetimes the gauge connection vanishes as well. This proves the first part of Theorem 4 in the introduction.

4.3. **Massive IIA supergravity.** Although Romans' massive IIA supergravity [**?**] does admit supersymmetric solutions, e.g., the D8-brane [**?**, **?**], it has no vacua for nonzero mass parameter. This can be easily seen by investigating the dilatino Killing spinor equation of the theory:

$$\begin{aligned} \Big(\partial_M \phi \Gamma^M + \tfrac{5}{4} m e^{\frac{5}{4}\phi} - \tfrac{3}{4} m e^{\frac{3}{4}\phi} B_{MN} \Gamma^{MN} \Gamma_{11} \\ - \tfrac{1}{6} e^{-\frac{1}{2}\phi} H_{MNR} \Gamma^{MNR} \Gamma_{11} + \tfrac{1}{48} e^{\frac{1}{4}\phi} F_{MNRP} \Gamma^{MNRP} \Big) \varepsilon = 0 \ , \end{aligned} \tag{69}$$

where B is the two-form gauge potential, $H = dB$, $F = dC + mB \wedge B$, ϕ is the dilaton and m is the cosmological constant. For maximal supersymmetry every term in the above Killing spinor equation must vanish separately. The first term implies that ϕ is constant, the third term implies that $B = 0$ and so $H = 0$, the last term implies that $F = 0$. However the second term cannot be made to vanish because m is nonzero. In summary, this proves Theorem 5 in the introduction. Note that it is not straightforward to take the limit $m \to 0$ in massive IIA to recover the usual IIA supergravity. This can only be done after appropriate redefinitions of the fields.

Acknowledgments

This work was completed while JMF was visiting the IHÉS and it is his pleasure to thank them and in particular Jean-Pierre Bourguignon for the invitation, their support and for providing an ideal environment in which to do research.

JMF is a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme, and his research is partially supported by the EPSRC grant GR/R62694/01.

The research of GP is partially supported by the PPARC grants PPA/G/S/1998/00613 and PPA/G/O/2000/00451 and by the European grant HPRN-2000-00122.

School of Mathematics, University of Edinburgh, Scotland, UK
E-mail address: `j.m.figueroa@ed.ac.uk`

Department of Mathematics, King's College, London, England, UK
E-mail address: `gpapas@mth.kcl.ac.uk`